

Disproving Hibi's Conjecture with CoCoA

or

Projective Curves with bad Hilbert Functions

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Introduction.

In this paper we show how to combine different techniques from Commutative Algebra and a systematic use of a Computer Algebra System (in our case mainly CoCoA (see [G-N] and [A-G-N])) in order to explicitly construct Cohen-Macaulay domains, which are standard k -algebras and whose Hilbert function is “bad”. In particular we disprove a well-known conjecture by Hibi.

To be more precise, we recall that a ring A is called a *standard k -algebra*, or simply *standard*, if k is a field and A is a finitely generated k -algebra, which is generated by its forms of degree 1 (see [S₁]). To every such a ring A a numerical function H_A is associated, namely the function $H_A : \mathbb{N} \longrightarrow \mathbb{N}$, which is defined by $H_A(r) := \dim_k A_r$ for every $r \in \mathbb{N}$. Here it should be noted that A can be represented as the quotient of $R := k[X_0, \dots, X_n]$ modulo a homogeneous ideal I , hence $\dim_k A_r \leq \dim_k R_r = \binom{n+r}{r}$. Such a function is called the *Hilbert function* of A . It is well-known (see [A-M]) that H_A can be encoded in the power series $\mathcal{P}_A(\lambda) := \sum_r H_A(r)\lambda^r \in \mathbb{Z}[[\lambda]]$, which is called the Hilbert-Poincaré series (or simply the Poincaré series) of A . This series is rational of type $\mathcal{P}_A(\lambda) = \frac{Q_A(\lambda)}{(1-\lambda)^d}$, with $Q_A(1) \neq 0$; moreover $Q_A(\lambda) = \sum h_i(A)\lambda^i \in \mathbb{Z}[\lambda]$ and if δ is its degree, then the integral vector $\mathbf{h}(A) := (h_0(A), h_1(A), \dots, h_\delta(A))$ is called the *h -vector of A* . It turns out that all the information of H_A can be encoded in $(\mathbf{h}(A), d)$; in particular d is the dimension and $\sum_i h_i(A) = Q_A(1)$ is the multiplicity

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of A . An efficient algorithm for the computation of $\mathcal{P}_A(\lambda)$ is described in [B-C-R] and implemented in CoCoA.

In the paper [Hi] Hibi defines an h -vector $(h_0, h_1, \dots, h_\delta)$ to be *flawless* if

- i) $h_0 \leq h_1 \leq \dots \leq h_{\lfloor \delta/2 \rfloor}$ and
- ii) $h_i \leq h_{\delta-i}$ for every i such that $0 \leq i \leq \lfloor \delta/2 \rfloor$

and he states the following

CONJECTURE: The h -vector $\mathbf{h}(A) := (h_0(A), h_1(A), \dots, h_\delta(A))$ of a standard Cohen-Macaulay domain is flawless.

The conjecture was supported by some empirical evidence and the fact that the statement is true under some additional hypotheses (see [Hi], Theorem 3.1).

The main goal of this paper is to construct *explicit examples of standard Cohen-Macaulay domains, whose h -vector is not flawless*. To this end we use a technique introduced by Galligo in [G], which yields sets of points in the projective space with the Uniform Position Property. A good deal of freedom in choosing equations allows us to construct projective coordinate rings of sets of points, whose h -vector “has a flaw”. Then we lift these sets to irreducible reduced rational projective curves in $\mathbb{P}_{\mathbb{C}}^4$, which turn out to be projectively Cohen-Macaulay. Their coordinate rings are the desired counterexamples.

This paper is largely inspired by the work of Galligo (see [G]) and by the calculations of some Galois groups, which were shown to the second author by G. Scheja, during a short visit to the University of Tübingen in June 1991. To both we are largely indebted.

§1. Preliminaries.

In this section we recall all the definitions and results, which we need later.

Definition. Let k be an infinite field, A a standard k -algebra, i.e. a graded k -algebra which is finitely generated by its linear forms and let $\mathcal{P}_A(\lambda) = \frac{Q_A(\lambda)}{(1-\lambda)^d}$, with $Q_A(1) \neq 0$, be its Poincaré series. Let $\delta := \deg(Q_A(\lambda))$ and $Q_A(\lambda) := \sum_{i=0}^{\delta} h_i(A)\lambda^i$. Then $\mathbf{h}(A) := (h_0(A), h_1(A), \dots, h_{\delta}(A))$ is called the h -vector of A . Sometimes it is denoted by $(h_0, h_1, \dots, h_{\delta})$, if there is no ambiguity about A .

The following facts are part of the folklore and are recalled only for the sake of completeness. The non explained terminology is part of the basic literature in Commutative Algebra and Algebraic Geometry (see for instance [A-M] and [Hart]).

Lemma 1.1. *Let $k \subset F$ be fields, A a standard k -algebra, $A_F := A \otimes_k F$. Then $\mathbf{h}(A_F) = \mathbf{h}(A)$.*

Proof. Indeed $\mathcal{P}_{A_F}(\lambda) = \mathcal{P}_A(\lambda)$ •

Proposition 1.2. *Let k be an infinite field and A a standard k -algebra of dimension d . Then there exist d linear forms L_1, \dots, L_d , such that $\dim(A/(L_1, \dots, L_d)) = 0$. If moreover A is Cohen-Macaulay (C-M), then L_1, \dots, L_d is a regular sequence in A .*

Corollary 1.3. *Let A be a Cohen-Macaulay standard k -algebra over a field k , let L_1, \dots, L_d be a maximal regular sequence of linear forms in A and denote by $B := A/(L_1, \dots, L_d)$. Then $\mathbf{h}(A) = \mathbf{h}(B)$.*

Proof. Indeed $\mathcal{P}_B(\lambda) = (1-\lambda)^d \mathcal{P}_A(\lambda)$ •

Let now $S := k[X_0, X_1, \dots, X_n]$, \mathfrak{M} a maximal homogeneous relevant ideal in S and $K := K_0(S/\mathfrak{M})$ its associated field, i.e. the field of homogeneous fractions of degree 0 of S/\mathfrak{M} . The scheme $\text{Proj}(S/\mathfrak{M})$ has a unique point, whose associated local ring is $K_0(S/\mathfrak{M})$. If $X_0 \notin \mathfrak{M}$, then we can dehomogenize \mathfrak{M} with respect to X_0 and we get a maximal ideal \mathfrak{m} in $R := k[X_1, \dots, X_n]$ (this can be done by putting $X_0 = 1$). It

turns out that $K \cong R/\mathfrak{m}$. Moreover every generic linear change of coordinates yields the following shape of \mathfrak{m}

$$\mathfrak{m} = (f(X_1), X_2 - g_2(X_1), X_3 - g_3(X_1, X_2), \dots, X_n - g_n(X_1, \dots, X_{n-1}))$$

It is clear that $K \cong k[X]/(f(X))$, hence $\deg(f(X)) = \dim_k K$.

Definition. In the above described situation we say that $f(X)$ represents \mathfrak{M} .

Definition. Let \mathfrak{M} be a maximal homogeneous relevant ideal in S , K its associated field and $d = \dim_k K$. We say that \mathfrak{M} is G -symmetric if the Galois group $\text{Gal}_k(K)$ is the full symmetric group Σ_d .

Corollary 1.4. *Let \mathfrak{M} be a maximal homogeneous relevant ideal in S , K its associated field and $f(X)$ a polynomial of degree d representing \mathfrak{M} . Then \mathfrak{M} is G -symmetric if and only if $\text{Gal}_k(f(X)) = \Sigma_d$*

Now we recall a well-known criterion

Theorem 1.5. *Let $f(X) \in \mathbb{Z}[X]$, $d := \deg(f(X))$ and assume that*

- a) $f(X)$ is irreducible
- b) *There exist two prime numbers p_1, p_2 such that, if we denote by $f_i(X)$ the residue classes of $f(X)$ modulo p_i , $i = 1, 2$, then*
 - b₁) $f_1(X)$ decomposes as the product of a linear factor and an irreducible factor of degree $d - 1$.
 - b₂) $f_2(X)$ decomposes as the product of an irreducible factor of degree 2 and irreducible factors of odd degrees.

Then $\text{Gal}_{\mathbb{Q}}(f(X)) = \Sigma_d$.

Proof. See [W] and [S-S] •

Definition. Let F be an infinite field and let $E \subset \mathbb{P}_F^n$ be a finite set of reduced points. Let I be the homogeneous ideal of $F[X_0, \dots, X_n]$ which defines E . We say that E is

G -symmetric if there exists a subfield $k \subseteq F$ and a maximal homogeneous relevant ideal \mathfrak{M} in $S := k[X_0, \dots, X_n]$ such that $I = \mathfrak{M}F[X_0, \dots, X_n]$ and $\text{Gal}_k(K_0(S/\mathfrak{M})) = \Sigma_d$

Theorem 1.6. *Let C be a projective irreducible reduced curve in $\mathbb{P}_{\mathbb{C}}^n$. Then the generic hyperplane section of C is a G -symmetric set.*

Proof. See [G], Proposition 13. The proof given for a smooth curve in \mathbb{P}^3 works as well in general •

Theorem 1.7. *Let $E \subset \mathbb{P}_F^n$ be a G -symmetric set of points and $A := F[X_0, \dots, X_n]/I$ its coordinate ring. Let $\mathbf{h}(A) := (h_0, h_1, \dots, h_\delta)$ be the h -vector of A . Then the following inequalities hold:*

$$h_0 + h_1 + \dots + h_i \leq h_{\delta-i} + \dots + h_{\delta-1} + 1$$

for every $i = 1, \dots, [\delta/2]$

Proof. It is proved in [G] that if E is G -symmetric, then it has the Uniform Position Property (UPP), i.e. the Hilbert function of its subsets depends only on their cardinality. Now if E is UPP, then every subset has the Cayley Bacharach (CB) property (see [G-K-R] for definitions and properties), and the conclusion follows again by [G-K-R] •

§2. The construction of the counterexamples.

Now we are ready to use the above described machinery in order to produce standard Cohen Macaulay domains with bad Hilbert functions. All the following computations have been done using CoCoA 1.7b on a Macintosh.

The first step is to produce polynomials $f(X) \in \mathbb{Z}[X]$ of degree d , such that $\text{Gal}_{\mathbb{Q}}(f(X)) = \Sigma_d$. This is not difficult, since the "generic" one has this property; however we want polynomials, which are not too dense, since we want to use them to make further computations.

Lemma 2.1. *The polynomial $f(X) := X^{18} - X - 1$ is such that $\text{Gal}_{\mathbb{Q}}(f(X)) = \Sigma_{18}$.*

Proof. We compute the factorization of $f(X)$ in $\mathbb{Z}[X]$ and then the factorization of its classes modulo successive primes. In this case we are particularly lucky, since we find that

a) $f(X)$ is irreducible

b₁) The complete factorization of $f(X)$ modulo 3 is $(X^2 - X - 1)(X^{13} - X^{12} + X^{11} + X^8 - X^7 - X^6 - X^5 + X^4 + X^3 - X^2 + 1)(X^3 - X^2 + 1)$

b₂) The complete factorization of $f(X)$ modulo 5 is $(X + 2)(X^{17} - 2X^{16} - X^{15} + 2X^{14} + X^{13} - 2X^{12} - X^{11} + 2X^{10} + X^9 - 2X^8 - X^7 + 2X^6 + X^5 - 2X^4 - X^3 + 2X^2 + X + 2)$

The conclusion follows from Theorem 1.5 •

Corollary 2.2. *Let I be the ideal of $\mathbb{Q}[X, Y, Z]$ generated by $(X^{18} - X - 1, Y - g(X), Z - h(X, Y))$ with $g(X) \in \mathbb{Q}[X]$ and $h(X, Y) \in \mathbb{Q}[X, Y]$. Let $\mathfrak{M} := {}^h I$ i.e the homogeneization of I with respect to a new indeterminate W . Then \mathfrak{M} is a G -symmetric maximal relevant ideal in $\mathbb{Q}[X, Y, Z, W]$.*

Proof. Clearly $\mathbb{Q}[X, Y, Z]/I \cong \mathbb{Q}[X]/(X^{18} - X - 1)$ and this is a field, hence I is a maximal ideal of $\mathbb{Q}[X, Y, Z]$. Consequently \mathfrak{M} is a maximal relevant ideal in $\mathbb{Q}[X, Y, Z, W]$. The conclusion follows from Corollary 1.4 and Lemma 2.1 •

Now the homogeneization of I is computed via a Gröbner basis computation with respect to an ordering which is degree-compatible and the leading term ideal of I and of hI are generated by the same elements, hence they have the same h -vector. The key point is now that we are *totally free* in the choice of $g(X)$ and $h(X, Y)$. We use again CoCoA and again we are lucky, because we do not need many experiments. Namely

Example 2.3. *Let I be the ideal of $\mathbb{Q}[X, Y, Z]$ generated by $(X^{18} - X - 1, Y - X^3, Z - XY)$ and let \mathfrak{M} be its homogeneization with respect to the new indeterminate W . Then $A := \mathbb{Q}[X, Y, Z, W]/\mathfrak{M}$ is a standard \mathbb{Q} -algebra, which is a Cohen-Macaulay domain and whose h -vector is $(1, 3, 5, 4, 4, 1)$. It satisfies the inequalities of Theorem 1.7, but it is not flawless.*

Proof. Let $A := \mathbb{Q}[X, Y, Z, W]/\mathfrak{M}$. The computation shows that $\mathfrak{M} = (XY - ZW, X^3 - YW^2, X^2Z - Y^2W, Y^3 - XZ^2, Y^2Z^3 - XW^4 - W^5, YZ^4 - X^2W^3 - XW^4, Z^5 - X^2W^3 - YW^4)$ and that $\mathcal{P}_A(\lambda) = \frac{(1+3\lambda+5\lambda^2+4\lambda^3+4\lambda^4+\lambda^5)}{(1-\lambda)}$. Moreover A is a domain and it is Cohen-Macaulay, since it is 1-dimensional and W is a non zero divisor modulo \mathfrak{M} •

This is already a counterexample to Hibi's conjecture!

However the fact that A is a domain heavily relies on the special ground field. Namely if we replace \mathbb{Q} with \mathbb{C} (it suffices to replace it with the decomposition field of $f(X)$), then $\mathbb{C}[X, Y, Z, W]/\mathfrak{M}\mathbb{C}[X, Y, Z, W]$ is a reduced \mathbb{C} -algebra, hence it is the coordinate ring of a G -symmetric set of 18 points in $\mathbb{P}_{\mathbb{C}}^3$. Its h -vector is still the same (see Lemma 1.1), but it is no more a domain.

So the final part is devoted to find a standard algebra whose h -vector is not flawless and which is a “geometric” domain, i.e the fact that it is a domain is not affected by any extension of the base field. The idea is to “lift” our previous example to an irreducible reduced curve in $\mathbb{P}_{\mathbb{C}}^4$. A small deformation of our equation $f(X)$ does the trick. Namely

Example 2.4. *Let \mathfrak{p} be the ideal of $\mathbb{C}[X, Y, Z, T]$ generated by $(X^{18} - X - 1 - T, Y -$*

$X^3, Z - XY)$ and let \mathfrak{P} be its homogeneization with respect to the new indeterminate W . Then $A := \mathbb{C}[X, Y, Z, T, W]/\mathfrak{P}$ is a standard \mathbb{C} -algebra, which is a Cohen-Macaulay domain and whose h -vector is $(1, 3, 5, 4, 4, 1)$, hence it is not flawless.

Proof. It is clear that $\mathbb{C}[X, Y, Z, T]/\mathfrak{p} \cong \mathbb{C}[X]$, hence \mathfrak{p} is a prime ideal, hence \mathfrak{P} is a prime ideal. It defines a projective rational curve in $\mathbb{P}_{\mathbb{C}}^4$. The actual computation yields the following minimal system of generators for \mathfrak{P} .

$$\mathfrak{P} = (XY - ZW, \quad X^3 - YW^2, \quad X^2Z - Y^2W, \quad Y^3 - XZ^2, \quad Y^2Z^3 + XW^4 + TW^4 + W^5, \quad YZ^4 + X^2W^3 + XTW^3 + XW^4, \quad Z^5 + X^2TW^2 + X^2W^3 + YW^4).$$

The computation of the Poincaré series yields $\mathcal{P}_A(\lambda) = \frac{(1+3\lambda+5\lambda^2+4\lambda^3+4\lambda^4+\lambda^5)}{(1-\lambda)^2}$. It remains to prove that A is a Cohen-Macaulay ring. For, it is enough to show that W, T is a regular sequence *mod* \mathfrak{P} . Indeed W is a non zero divisor *mod* \mathfrak{P} , since it is the homogeneizing indeterminate. If we compute the quotient modulo W , we get $\mathbb{C}[X, Y, Z, T]/(XY, X^3, X^2Z, Y^3 - XZ^2, Y^2Z^3, YZ^4, Z^5)$. Hence clearly T does not divide zero •

We conclude the paper with some remarks.

Remark. Example 2.4 disproves Hibi's Conjecture. So it is interesting to check that it does not fit with the special class described by Hibi in [Hi] Theorem 3.1. There it is required that the associated order ideal of monomials is pure. Without going too much in to the details, we check that in our case the associated order ideal of monomials is

$$\{1, X, Y, Z, X^2, XZ, Y^2, YZ, Z^2, XZ^2, Y^2Z, YZ^2, Z^3, XZ^3, Y^2Z^2, YZ^3, Z^4, XZ^4\}$$

Its maximal elements are X^2, Y^2Z^2, YZ^3, XZ^4 whose degrees are 2, 4, 4, 5; therefore the order ideal of monomials is not pure.

Remark. One can construct similar examples to 2.4. For instance if we consider the ideals $(X^{22} - X - 1 - T, Y - X^3, Z - XY)$, $(X^{26} - X - 1 - T, Y - X^3, Z - XY)$, $(X^{30} - X - 1 - T, Y - X^3, Z - XY)$ and carry over the same construction as in 4.2, we see that:

the Galois group of $X^{22} - X - 1$ is Σ_{22} . The primes who do the trick as in Lemma 2.1 are 29 and 107. The corresponding h -vector is $(1, 3, 5, 4, 4, 4, 1)$.

The Galois group of $X^{26} - X - 1$ is Σ_{26} . The primes who do the trick as in Lemma 2.1 are 19 and 67. The corresponding h -vector is $(1, 3, 5, 4, 4, 4, 4, 1)$.

The Galois group of $X^{30} - X - 1$ is Σ_{30} . The primes who do the trick as in Lemma 2.1 are 5 and 53. The corresponding h -vector is $(1, 3, 5, 4, 4, 4, 4, 4, 1)$.

All of them are not flawless.

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